Laminar Hypersonic Trail in the Expansion-Conduction Region

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The usual procedure in calculating the cooling process in a wake behind a blunt object is to assume a region of pure expansion up to a distance where the pressure has reached its ambient value, followed by a region where the mechanism of pure heat conduction is operative. In the present paper both mechanisms are assumed to be valid simultaneously, and the result is compared with previous calculations. The following criterion is established: the minimum radius of a hemisphere-cylinder configuration, above which a simultaneous conduction-expansion calculation is not needed, is given by the approximation

$$(Re)_{\min} = \frac{\rho_\infty U_\infty r_{\min}}{\mu_\infty} \simeq \frac{6000}{M^2}\left(\frac{h}{h_\infty}\right)^2_{\min}$$

where $(h/h_\infty)_{\min}$ is the nondimensional value of the enthalpy at the axis of the wake below which the two methods of computation give the same result, and M is the flight Mach number.

Nomenclature

A = constant defined in Eq. (13)
B = constant defined in Eq. (17)
b = constant defined in Eq. (5)
h = enthalpy
M = Mach number calculated at the freestream
m = exponent determining the viscosity law [Eq. (11)]
Pr = Prandtl number
p = pressure
R = transformed radial distance defined in Eq. (9)
Re = Reynolds number
r = radial distance
r_0 = radius of spherical body
U_∞ = velocity in the freestream
u = velocity in the wake
X = nondimensional distance in the x direction defined in Eq. (9)
x = direction of the main flow
x_i = distance behind the stagnation point from which the cooling process is described
x_0 = distance behind the stagnation point where the ambient pressure has been reached
α = enthalpy ratio defined in Eq. (6)
β = Gaussian-like depth defined in Eq. (6)
γ = ratio of specific heats
μ = coefficient of viscosity
ρ = mass density
ϕ = auxiliary function defined in Eq. (14)
∞ = subscript referring to the freestream

I. Introduction

TRAILS of bodies moving at hypersonic speeds are of interest to meteor physicists and those concerned with the problem describing man-made objects re-entering the earth's dense atmosphere. Several papers have appeared recently which attempt to give an account of the different phenomena involved in the production of the trail and their possible use in predicting the mass, geometry, and flight characteristics of the object causing it.[1-3]

For a blunt body, the fluid is compressed through the bow-shock wave and expanded until it is recompressed again through any existing flares and the so-called trailing shock, after which further expansion takes place until the ambient pressure is reached at all streamlines at a distance, say, $x = x_0$. This distance can be calculated as indicated in Ref. 1 by using the blast-wave analogy, notwithstanding the fact that this theory allows only the compression due to the bow-shock wave; all other compressions, including the one due to the trailing shock, have been shown[1, 2] ta create a small amount of irreversibility when compared to the one created by the bow-shock wave, so that if one is interested in calculating the flow at a distance far downstream, its influence can be neglected in a first approximation. One thus is left essentially with a hemisphere-cylinder geometry over which, as first suggested in Ref. 2, the flow can be calculated by standard numerical techniques, through the method of characteristics using real gas properties.

From the second-order blast-wave theory of Sakurai[4] and for a drag coefficient equal to one, the pressure decays along the distance x according to the formula

$$p/p_\infty = 0.1330[M^2/(x/r_0)] + 0.405 \qquad (1)$$

An estimate of the distance $x = x_0$ over which the pressure is almost ambient can be obtained from Eq. (1) by making $p/p_\infty = 1$; in this fashion, one finds

$$(x/r_0)_{\exp} \simeq M^2/4.5 \qquad (2)$$

A comparison in Ref. 1 shows that this approximation is verified by the numerical integrations of Ref. 2. It should be noted, however, that, although Eq. (1) provides a good approximation for small distances downstream, as $x/r_0 \to \infty$ it yields $p/p_\infty \to 0.405$, an obviously erroneous result. Whereas the pressure becomes almost ambient at the distance $x = x_0$, the temperature still retains a maximum value at the axis of symmetry, and the velocity at the same point exhibits a maximum deficiency. It becomes apparent then, that, because of the existence of these gradients and since further cooling is not possible through the mechanism of expansion, cooling will occur through thermal conduction.

All detailed calculations reported so far in the literature (e.g., Refs. 1–3) assume pure expansion up to the length x_0 and pure conduction for lengths higher than x_0. Feldman,[2] after a remark made by R. Goulard, points out that the foregoing procedure should be modified for small bodies for which the mechanisms of cooling by expansion and conduction are simultaneously active.† The present paper is an attempt to

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† *Added in proof:* In a recent paper,[9] Steiger and Bloom have obtained similar solutions for several cases of linearized free mixing with streamwise pressure gradients.

provide a solution in a closed form over a hemisphere-cylinder for such a case. The results of the analysis then could be used to find the exact conditions under which the method of separate expansion and conduction is legitimate.

First one finds, through a simple argument, the nondimensional parameter that will provide a measure of the importance of the two mechanisms of cooling. From Eq. (2), one has $(x)_{\exp} \sim M^2 r_0$. The length $(x)_{\exp}$ is a measure of the characteristic length needed for cooling to ambient pressure by pure expansion. The equivalent length $(x)_{\text{cond}}$ for conduction can be estimated by equating the order of magnitude of the convective and conduction terms in the equation of energy conservation. For Prandtl number 1, this length is $(x)_{\text{cond}} \sim (Re)\, r_0$, where Re is the Reynolds number. Comparison of the two lengths yields

$$(x)_{\exp}/(x)_{\text{cond}} \sim M^2/Re \tag{3}$$

An exact analysis will provide the numerical constant for the number M^2/Re above which there will be need for a detailed conduction-expansion calculation.

II. Analysis

The following assumptions will be made:

1) The pressure decays downstream according to the formula

$$p/p_\infty = 1 + [b(M)/(x/r_0)] \tag{4}$$

In Eq. (4), $b(M)$ is a suitable function of the freestream Mach number which can be chosen for best fit with experimental data. The form of Eq. (4) is suggested by the blast-wave theory,[4] and it has the desirable property for $x/x_0 \to \infty$ to make $p/p_\infty \to 1.0$ in variance with Eq. (1). For $x/x_0 > 5$, a good correlation with experimental data[5] can be obtained if

$$b(M) = [4(M^2 - 44.6)]/30 \tag{5}$$

The general theory, however, is not restricted by the form of Eq. (5).

2) The enthalpy profiles are similar in the sense of the following equation:

$$h/h_{x_i} = \alpha(x) f[R/\beta(x)] \tag{6}$$

with $f(0) = 1$, where $\alpha(x)$ is the local maximum nondimensional enthalpy and $\beta(x)$ a corresponding Gaussian-like depth. The assumption also is made that in the neighborhood of the axis the function f behaves like a parabola. From conservation of total energy crossing any plane perpendicular to the axis, one obtains[1, 2]

$$\alpha(x)\beta(x) = \alpha(x_i)\beta(x_i) = \text{const} \tag{7}$$

The distance x_i/r_0 could be taken conveniently as equal to, say, five, above which the Gaussian character of the enthalpy is established (see Ref. 2) for a sphere-cylinder model.

3) For the calculation of the convective terms in the energy equation, use a constant velocity‡ $u/U_\infty \simeq 0.8$. The energy equation is

$$\rho u \frac{\partial h}{\partial x} = u \frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu}{Pr}\, r \frac{\partial h}{\partial r}\right) \tag{8}$$

Introducing the coordinates

$$X = \frac{x/r_0}{Re} = \frac{(x/r_0)\,\mu_\infty}{\rho_\infty U_\infty r_0} \qquad R^2 = \int_0^r \frac{\rho_\infty}{\rho}\, d\, r^2 \tag{9}$$

‡ In reality, study of the momentum equation shows that the full value of $u/U_\infty = 1$ is recovered at the same distance where $h/h_\infty = 1$. The value of 0.8 corresponds at $x = x_0$ and is almost independent of Mach number. This was shown numerically in Ref. 2. In Appendix A of Ref. 8, it is shown analytically that $u/U_\infty \simeq [1 - M^{-1/3}]^{1/2}$. For M between 10 and 35, the foregoing gives u/U_∞ between 0.73 and 0.83.

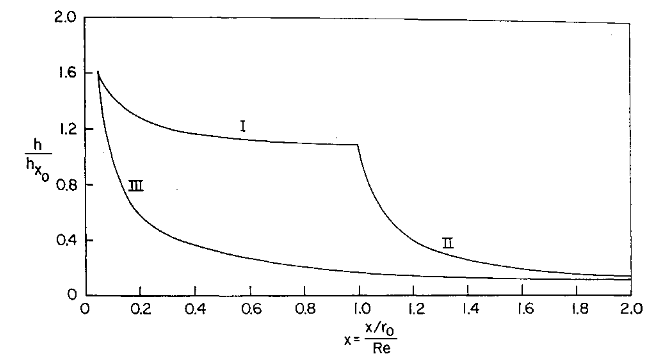

Fig. 1 Cooling of the stagnation streamline by expansion (curve I) and then conduction (curve II) compared with simultaneous consideration of expansion and conduction (curve III) for $Re = 100$ and $M = 20$

and satisfying the energy equation at the axis where $R \to 0$, one finds

$$\frac{d\alpha(X)}{dX} - \frac{\rho_\infty}{\rho}\frac{p_\infty}{h_\infty \rho_\infty}\frac{d(p/p_\infty)}{dX} = \frac{4U_\infty \mu}{Pr\, u\, \mu_\infty \alpha(x_i)\beta(x_i)}\alpha^2(X) \tag{10}$$

For the estimation of the contribution of the density factor in the pressure term only, assume ideal-gas relations with appropriate γ's taken from Ref. 6. Furthermore, in the fashion of Ref. 1, assume a viscosity law as follows:§

$$\mu/\mu_i = (h/h_{x_i})^m \tag{11}$$

where m is of the order of 0.25. With the foregoing substitutions, the energy equation becomes

$$\frac{d\alpha(X)}{dX} - \left(\frac{\gamma - 1}{\gamma}\right)\alpha(X)\frac{d(\ln p/p_\infty)}{dX} + A\, \alpha^{2+m}(X) = 0 \tag{12}$$

where

$$A = \frac{4\mu_i}{Pr \cdot \mu_\infty \beta(x_i)}\left(\frac{U_\infty}{u}\right) \tag{13}$$

Equation (13) is a generalized Riccati equation in its simple form, with the right-hand member zero and the last term raised to the $2 + m$ power rather than two. It can be solved by using the following substitution:

$$\alpha \equiv \left[\frac{d\phi/dX}{A\phi(m+1)}\right]^{1/(1+m)} \tag{14}$$

Substitution of Eq. (14) yields

$$\frac{d^2\phi}{dX^2} - (m+1)\frac{(\gamma - 1)}{\gamma}\frac{d\phi}{dX} \cdot \frac{d\ln(p/p_\infty)}{dX} = 0 \tag{15}$$

One quadrature yields

$$\frac{d\phi}{dX} = C\left(\frac{p}{p_\infty}\right)\frac{(m+1)(\gamma-1)}{\gamma} = C\left(1 + \frac{B(M)}{X}\right)^{\frac{[(m+1)(\gamma-1)]}{\gamma}} \tag{16}$$

where

$$B(M) \equiv b(M)/Re \tag{17}$$

and C is a constant of integration.

Equation (16) is general and can be integrated for any suitable values for γ and m and any function $B(M)$. From Ref. 6, take $\gamma/(\gamma - 1) \approx 5$, and from Ref. 1, $m \approx 0.25$;

§ The viscosity is introduced artificially here in order to calculate the variable thermal conductivity through the Prandtl number.

hence, $\gamma/(m+1)(\gamma-1) \simeq 4$. One more integration and application of the boundary condition yields

$$\alpha^{-(1+m)} = \left[\frac{1+(B/X_i)}{1+(B/X)}\right]^{1/4} + A(1+m)X - A(1+m)\left[\frac{1+(B/X_i)}{1+(B/X)}\right]^{1/4} X_i + \frac{AB(1+m)}{4[1+(B/X)]^{1/4}} \ln\left|\frac{\{1+[1+(B/X)]^{1/4}\}}{\{1-[1+(B/X)]^{1/4}\}}\frac{\{1-[1+(B/X_i)]^{1/4}\}}{\{1+[1+(B/X_i)]^{1/4}\}}\right| + \frac{AB(1+m)}{4[1+(B/X)]^{1/4}}\left\{\tan^{-1}\left(1+\frac{B}{X}\right)^{1/4} - \tan^{-1}\left(1+\frac{B}{X_i}\right)^{1/4}\right\} \quad (18)$$

It is easy to see that‖ when $B = 0$ and $X_i = 0$ the solution reduces to the one corresponding to pure conduction:[1]

$$\alpha^{(1+m)} = 1/[1 + A(1+m)X] \quad (19)$$

The same solution is obtained when B is small but X large.

The cooling process along the stagnation streamline according to Eq. (18) will be compared now with the one obtained by first expanding isentropically to a distance x_0, where the pressure has reached approximately its ambient value, with further cooling occurring by the simple conduction model of Eq. (19). For x_0, the estimate provided by Eq. (2) is used, because all previous papers in which the expansion and conduction mechanisms operate independently have used this approximation. For a Mach number of the order of 20, $(x/r_0)_{\rm exp}$ is about 100. For the same Mach number from Eq. (5), one obtains $b \approx 50$. On the other hand, the order of magnitude of the constant A is about 10.# Choose $x_i/r_0 = 5$, $m = 0.25$, and $\gamma/(\gamma-1) = 5$. Curve I in Fig. 1 shows, for a Reynolds number of 100, the cooling process along the center line for a pure isentropic expansion. This curve is given by the relation

$$\frac{h}{h_{x_0}} = \left(1 + \frac{b}{x/r_0}\right)^{(\gamma-1)/\gamma} \quad (20)$$

For the numerical case under discussion, curve I of Fig. 2 starts at the point $X_i = 0.05$, where from Eq. (20) one finds $h/h_{x_0} \simeq 1.61$, and terminates at $X_{\rm exp} = (x/r_0)_{\rm exp}/Re = 1.0$, where $h/h_{x_0} \simeq 1.09$. Curve II is the result of Eq. (19) normalized to the value of $\alpha = h/h_{x_0}$ at the point $X_{\rm exp}$ of curve I.** Curve III is Eq. (18) normalized at the point $X = X_i$ of curve I. Figure 2 is a similar plot, the one difference being in the Reynolds number, which is now equal to 1000. Figures 1 and 2 also represent conditions for which the value of A differs from 10. It is easy to show by observation of Eq. (18) that for any other value of A it is enough to multiply the values of X and Re by $10/A$ and $A/10$, respectively. Figures 3 and 4 are the equivalent of Figs. 1 and 2 for a Mach number equal to 15, for which $b \simeq 25$ and $(x/r_0)_{\rm exp} \simeq 50$. Equation (18) gave almost the same numerical results as in the case of $M = 20$. This means that curves III in Figs. 1 and 3 corresponding to the same Reynolds number but different Mach number are almost the same after they both are normalized at the initial station $X = X_i$. The same is true for curves III in Figs. 2 and 4. Note that curves I and II lie closer to curve III for the smaller Mach number because the expansion length $(x/r_0)_{\rm exp}$ is smaller. Figures 2 and 4 show that curves II slightly undershoot curves III. This is to be expected, since for high X's curve III, based on Eq. (18), reduces to Eq. (19) multiplied by the ratio $(h/h_{x_0})_i/(h/h_{x_0})_{\rm exp}$, which is slightly higher than 1.

It is apparent that considerable difference exists between the two methods of calculation for small Reynolds numbers and distances close to the body. For a velocity of about 20,000 fps ($M_\infty \approx 20$) at an altitude of 100,000 ft, the Reynolds number is about 10^4 cm^{-1}, and it is obvious from Figs. 1 and 2 that even for a 1-mm radius there should be no difference between the two methods. For an altitude of 200,000 ft, the Reynolds number is about 300 cm^{-1}, and for a body 100 cm in diameter the simultaneous consideration of expansion and conduction is not warranted. However, if one tries to simulate actual re-entry with a pellet $\frac{2}{3}$ cm in diameter ($Re = 100$), say, in a ballistic range, then one sees from Fig. 1 that detailed calculation is necessary.

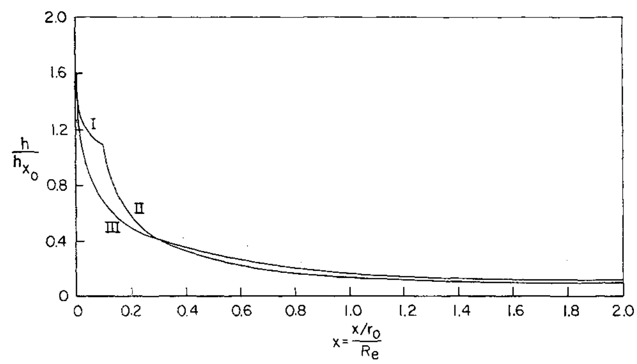

Fig. 2 Cooling of the stagnation streamline by expansion (curve I) and then conduction (curve II) compared with simultaneous consideration of expansion and conduction (curve III) for $Re = 1000$ and $M = 20$

‖ Note that B from its definition [Eq. (17)] and the approximation of Eq. (5) depends, for high Mach numbers, on the quantity M^2/Re alone, a result compatible with the nondimensional argument that led to Eq. (3).

Take as characteristic values $\mu_i/\mu_\infty \simeq 2$, $Pr = 1.0$, $\beta(x_i) = 1$, and $u/U_\infty, = 0.8$. Note that specific numbers are used for purposes of illustration. The interested reader may use any values for m, γ, A and any desired approximation for the function $b(M)$.

** Note that h_{x_0} denotes the value of the enthalpy of the stagnation streamline after complete isentropic expansion to the ambient pressure.

III. Conclusions

It is concluded that the parameter describing the relative importance of the expansion and conduction mechanisms is essentially†† the ratio M^2/Re. In physical language, this parameter is an expression of the fact that the characteristic lengths needed for expansion and conduction are of the same order. For a given flight Mach number, a series of figures similar to the ones already presented can be constructed for different Reynolds numbers. Each one of these figures yields a value for h/h_{x_0} below which separate computation of the expansion and conduction regions coincide with the simultaneous one. From a practical point of view, this value should be identified with the quantity $(h/h_{x_0})_{\rm min}$ corresponding to the smallest value of an observable that is measurable by a given instrument (such as electron concentration, optical or infrared radiation, etc.). An analysis of such figures has shown that in a good approximation $(h/h_{x_0})_{\rm min}{}^2$, as defined in the foregoing, is equal to 25 Re/M^2. The following criterion therefore can be stated: the minimum radius of a hemisphere-cylinder configuration in hypersonic flight above which a simultaneous expansion-conduction calculation is not needed is given by the approximation

$$(Re)_{\rm min} = \rho_\infty U_\infty r_{\rm min}/\mu_\infty \simeq 25M^2(h/h_{x_0})_{\rm min}{}^2 \quad (21)$$

†† Apart from the influence of Prandtl number and initial conditions as expressed by the quantities A and X_i.

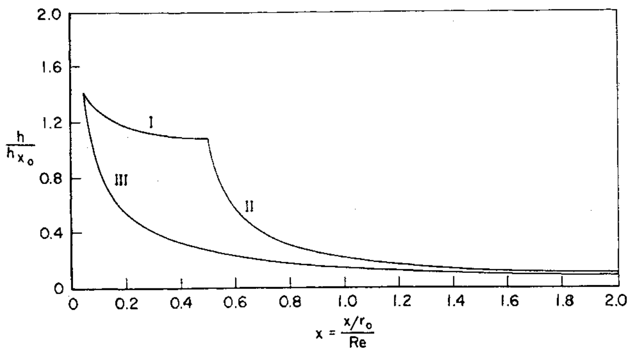

Fig. 3 Cooling of the stagnation streamline by expansion (curve I) and then conduction (curve II) compared with simultaneous consideration of expansion and conduction (curve III) for Re = 100 and M = 15

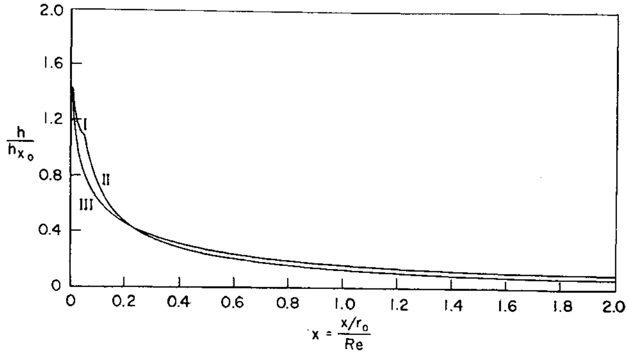

Fig. 4 Cooling of the stagnation streamline by expansion (curve I) and then conduction (curve II) compared with simultaneous consideration of expansion and conduction (curve III) for Re = 1000 and M = 15

For Mach numbers around 10 or higher, one may set the approximation

$$h_{x0}/h_\infty \simeq M^2/15 \qquad (22)$$

In this case Eq. (21) becomes, after rounding up the numerical coefficient,

$$(Re)_{\min} \simeq (6000/M^2)(h/h_\infty)_{\min}{}^2 \qquad (23)$$

It is concluded that simultaneous expansion and conduction becomes important for high altitudes (small ρ_∞), small bodies, and small flight velocities. It should be remarked that these are precisely the same conditions for which chemical relaxation effects become important.

Before closing, an example will be given. Feldman, on p. 445 of Ref. 2, reports that, for an altitude of 250,000 ft and velocity of 25,000 fps, the optical radiation in the pure conduction trail decays radically (one order of magnitude) in a length equivalent to $5r_0$ radii from the point $x = x_0$. Since under these conditions the distance x_0 is approximately equal to 150 radii, an object 30 cm in radius would be given an expansion-controlled trail equal to the conduction-controlled one, and hence a simultaneous calculation for expansion and conduction would be necessary.‡‡ From Ref. 2, at a distance from $x = x_0$ equal to $5r_0$ radii, the value $h/h_{x_0} = 0.7$ is estimated. For the altitude and Mach number under investigation, Eq. (21) gives $r_{\min} \simeq 75$ cm. This is the radius above which the two methods of calculation coincide below the value $h/h_{x_0} = 0.7$. On the other hand, if it is of interest to study the infrared radiation that, from Ref. 2, drops in the pure conduction trail much slower than the optical radiation (one estimates from Ref. 2 that, for an order of magnitude drop, $h/h_{x_0} \simeq 0.23$), one obtains from Eq. (21) $r_{\min} \simeq 8$ cm. Feldman's rough calculation of the same number is $r_{\min} \simeq 3$ cm.

‡‡ Feldman found also that at the radius of 30 cm the slope of the optical intensity at $x = x_0$ for the pure expansion and pure conduction trail are equal.[7]

References

1 Lykoudis, P. S., "Theory of ionized trails for bodies at hypersonic speeds," Rand Corp. Research Memo. RM-2682 (May 29, 1961); a condensed form of this paper can be found in *Proceedings of the 1961 Heat Transfer and Fluid Mechanics Institute* (Stanford University Press, Stanford, Calif., 1961), pp. 176–192.

2 Feldman, S., "Trails of axi-symmetric hypersonic blunt bodies flying through the atmosphere," J. Aerospace Sci. **28,** 433–448 (1961).

3 Goulard, M. and Goulard, R., "The aero-thermodynamics of re-entry trails," ARS Preprint 1145-60 (May 1960).

4 Lees, L. and Kubota, T., "Inviscid hypersonic flow over blunt-nosed slender bodies," J. Aeronaut. Sci. **24,** 195–197 (1957).

5 Mueller, J. and Close, W., "An investigation of induced-pressure phenomena on axially symmetric flow-aligned cylindrical models equipped with different non-shapes at free-stream Mach numbers from 15.6 to 21 in helium," NASA TN D-373 (May 1960).

6 Logan, J. G. and Treanor, C. E., "Polytropic exponents for air at high temperatures," J. Aeronaut. Sci. **24,** 467–468 (1957).

7 Feldman, S., private communication (June 22, 1962).

8 Lykoudis, P. S., "The growth of the hypersonic wake behind blunt and slender bodies," Rand Corp. Research Memo. RM-3270-PR (January 1963).

9 Steiger, M. H. and Bloom, M. H., "Linearized viscous free mixing with streamwise pressure gradients," Polytech. Inst. Brooklyn Rept. 777, OSR Grant 163 (March 1963).